J. Fluid Mech. (1978), *vol.* 89, *part* 1, *pp.* 191–200

Printed in Great Britain

A numerical study of the deformation and burst of a viscous drop in an extensional flow

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(Received 13 March 1978)

We study the deformation and conditions for breakup of a liquid drop of viscosity $\lambda\mu$ freely suspended in another liquid of viscosity μ with which it is immiscible and which is being sheared. The problem at zero Reynolds number is formulated exactly as an integral equation for the unknown surface velocity, which is shown to reduce to a particularly simple form when $\lambda = 1$. This equation is then solved numerically, for the case in which the impressed shear is a radially symmetric extensional flow, by an improved version of the technique used, for $\lambda = 0$, by Youngren & Acrivos (1976) so that we model the time-dependent distortion of an initially spherical drop. It is shown that, for a given λ, a steady shape is attained only if the dimensionless group $\Omega \equiv 4\pi G\mu a/\gamma$ lies below a critical value $\Omega_c(\lambda)$, where G refers to the strength of the shear field, a is the radius of the initial spherical drop and γ is the interfacial tension. On the other hand, when $\Omega > \Omega_c$ the drop extends indefinitely along its long axis. The numerical results for $\lambda = 0{\cdot}3$, $0{\cdot}5$, 1, 2, 10 and 100 are in good agreement with the predictions of the small deformation analysis by Taylor (1932) and Barthès-Biesel & Acrivos (1973) and, at the smaller λ, with those of slender-body theory (Taylor 1964; Acrivos & Lo 1978).

1. Introduction

When individual drops of one fluid are placed in another liquid which is undergoing shear, they will deform and, if the shear rate is sufficiently large, break up. There is thus a limiting size for a drop that will not burst in a shear flow of given strength. These phenomena have implications in a variety of seemingly diverse problems, such as the dispersion of one liquid phase into another, the rheology of emulsions, and tertiary oil recovery (Taylor 1934; Frankel & Acrivos 1970; Grace 1971; Barthès-Biesel & Acrivos 1973). Theoretical studies of drop deformation with negligible inertia have dealt primarily with the two limiting cases in which the deformation from a spherical shape is either small (Taylor 1932; Cox 1969; Barthès-Biesel & Acrivos 1973), or large, so that the drop is elongated and the methods of slender-body theory are applicable (Taylor 1964; Buckmaster 1972, 1973; Acrivos & Lo 1978). Exceptions are the work of Richardson (1968) and Buckmaster & Flaherty (1973), who employed complex-variable analysis to study two-dimensional drops. This approach, however, cannot be extended to the more realistic three-dimensional case.

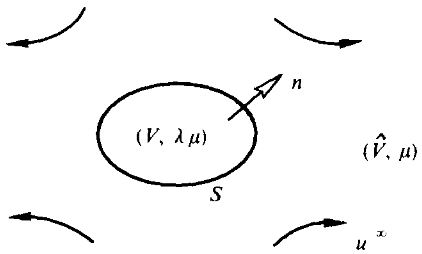

FIGURE 1. Schematic sketch for drop deformation.

Recently, Youngren & Acrivos (1976) developed a numerical technique to calculate the steady deformation of an inviscid drop placed in an extensional flow. The Stokes equations for the flow outside the drop were reformulated as an integral equation for the velocities and Stokeslets over the surface of the drop, and the drop shape was systematically adjusted until the condition of zero normal velocity was satisfied everywhere on the surface. The present paper presents an improved version of the integral formulation of Youngren & Acrivos (1976) which enables drops of arbitrary (non-zero) viscosity to be treated. The representation used is shown to generate a particularly simple expression for the surface velocities when the interior and exterior fluid viscosities are equal, a simplification that will be exploited for non-axisymmetric flows in a later paper. Here, however, we concentrate on the case of a uniaxial extensional flow and solve numerically for the time-dependent distortion of an initially spherical drop of arbitrary viscosity until it either attains a steady shape or starts to grow without limit. Our present results for the deformation as a function of flow strength at a given viscosity ratio, and for the variation with viscosity ratio of the critical flow strength required for breakup, compare favourably with the earlier predictions from the asymptotic analyses for small distortion (Barthès-Biesel & Acrivos 1973) and large distortion (Acrivos & Lo 1978).

2. Statement of problem and integral formulation

We consider a drop of viscosity $\lambda\mu$ which is immersed in a fluid of viscosity μ and which has interfacial surface tension γ. The fluid at infinity is made to flow with velocity $\mathbf{u}^\infty = \mathbf{E}.\mathbf{x}$ and the drop consequently deforms. Figure 1 shows the problem schematically. We consider a low Reynolds number deformation which is, therefore, quasi-static. Consequently $\mathbf{u}(\mathbf{x})$, the fluid velocity at each instant, is governed by the following equations:

$$\nabla.\mathbf{u} = 0 \quad \text{everywhere}, \quad \nabla.\boldsymbol{\sigma} = 0 \quad \text{for} \quad \mathbf{x}\notin S, \tag{1a,b}$$

$$\boldsymbol{\sigma} = \begin{cases} -p\mathbf{I} + \mu(\nabla\mathbf{u}+\nabla\mathbf{u}^{\mathrm{T}}) & \text{for} \quad \mathbf{x}\in\hat{V}, \\ -p\mathbf{I} + \lambda\mu(\nabla\mathbf{u}+\nabla\mathbf{u}^{\mathrm{T}}) & \text{for} \quad x\in V. \end{cases} \tag{1c}$$

The boundary conditions are

$$\mathbf{u}\rightarrow\mathbf{u}^\infty \quad \text{as} \quad |\mathbf{x}|\rightarrow\infty,$$

$$[\mathbf{u}]_S = 0, \quad [\boldsymbol{\sigma}.\mathbf{n}]_S = \gamma\mathbf{n}\nabla.\mathbf{n}, \tag{2b}$$

where $[\;]_S$ denotes the jump across the surface of the drop S from the outside to the inside, $\mathbf{n}$ is the outward normal and $\nabla.\mathbf{n}$ is the surface curvature. The solution of (1) and (2) gives the instantaneous velocity of every point on S, and the rate of deformation is determined by the normal component of this velocity.

Since the full solution of (1) with (2) contains very much more information than is needed to determine the deformation, a more convenient formulation of the problem is as an integral equation for the unknown surface velocities. The method is discussed by Ladyzhenskaya (1963) and was used by Youngren & Acrivos (1975, 1976) for solid particles and for droplets with $\lambda = 0$. We outline here the derivation of our basic equation for general λ.

Following Ladyzhenskaya (1969, chap. 3), we have for the exterior problem ($\mathbf{x} \in \hat{V}$)

$$u_i'(\mathbf{x}) + \int_{S_y} K_{ijk}(\mathbf{x}-\mathbf{y})\, u_j'(\mathbf{y})\, n_k(\mathbf{y})\, dS_y = -\frac{1}{8\pi\mu}\int_{S_y} J_{ij}(\mathbf{x}-\mathbf{y})\, \sigma_{jk}'(\mathbf{y})\, n_k(\mathbf{y})\, dS_y, \quad (3)$$

where $\mathbf{u}' \equiv \mathbf{u} - \mathbf{u}^\infty$ denotes the disturbance velocity, $\sigma_{ij}' \equiv \sigma_{ij} - \sigma_{ij}^\infty$ is the corresponding stress tensor arising from the disturbed flow, and

$$K_{ijk}(\mathbf{r}) \equiv -\frac{3}{4\pi}\frac{r_i r_j r_k}{r^5}, \quad J_{ij}(\mathbf{r}) \equiv \frac{\delta_{ij}}{r} + \frac{r_i r_j}{r^3}, \quad \mathbf{r} \equiv \mathbf{x} - \mathbf{y}. \quad (4)$$

However, since $\boldsymbol{\sigma}^\infty(\mathbf{x})$ and $\mathbf{u}^\infty(\mathbf{x})$ have no singularities except at infinity, application of the divergence theorem to (3) followed by integration over V easily leads to the result that, for any imposed flow $\mathbf{u}^\infty(\mathbf{x})$ satisfying the creeping-flow equations,

$$u_i(\mathbf{x}) + \int_{S_y} K_{ijk}(\mathbf{x}-\mathbf{y})\, u_j(\mathbf{y})\, n_k(\mathbf{y})\, dS_y = u_i^\infty(\mathbf{x}) - \frac{1}{8\pi\mu}\int_{S_y} J_{ij}(\mathbf{x}-\mathbf{y})\, \sigma_{jk}(\mathbf{y})\, n_k(\mathbf{y})\, dS_y \quad (5)$$

for $\mathbf{x} \in \hat{V}$. Then on letting $\mathbf{x} \to S$ from $\mathbf{x} \in \hat{V}$ we have, on account of the well-known jump condition (Ladyzhenskaya 1969, p. 57) for the first integral in (5),

$$\tfrac{1}{2}u_i(\mathbf{x}) + \int_{S_y} K_{ijk}(\mathbf{x}-\mathbf{y})\, u_j(\mathbf{y})\, n_k(\mathbf{y})\, dS_y = u_i^\infty(\mathbf{x}) - \frac{1}{8\pi\mu}\int_{S_y} J_{ij}(\mathbf{x}-\mathbf{y})\, \sigma_{jk}(\mathbf{y})\, n_k(\mathbf{y})\, dS_y \quad (6)$$

for $\mathbf{x} \in S$. Similarly, from the corresponding interior problem ($\mathbf{x} \in V$), we have

$$\tfrac{1}{2}u_i(\mathbf{x}) - \int_{S_y} K_{ijk}(\mathbf{x}-\mathbf{y})\, u_j(\mathbf{y})\, n_k(\mathbf{y})\, dS_y = \frac{1}{8\pi\lambda\mu}\int_{S_y} J_{ij}(\mathbf{x}-\mathbf{y})\, \sigma_{jk}(\mathbf{y})\, n_k(\mathbf{y})\, dS_y \quad (7)$$

for $\mathbf{x} \in S$. Hence, combining (6) and (7) and making use of (2), we conclude that for $\mathbf{x} \in S$

$$\tfrac{1}{2}(1+\lambda)\, u_i(\mathbf{x}) + (1-\lambda)\int_{S_y} K_{ijk}(\mathbf{x}-\mathbf{y})\, u_j(\mathbf{y})\, n_k(\mathbf{y})\, dS_y = u_i^\infty(\mathbf{x}) - \frac{\gamma}{8\pi\mu}\int_{S_y} J_{ij}(\mathbf{x}-\mathbf{y})\, n_j(\mathbf{y})\, \nabla\cdot\mathbf{n}\, dS_y. \quad (8)$$

We note in passing that when $\lambda = 1$ we obtain the particularly simple form

$$u_i(\mathbf{x}) = u_i^\infty(\mathbf{x}) - \frac{\gamma}{8\pi\mu}\int J_{ij}(\mathbf{x}-\mathbf{y})\, n_j(\mathbf{y})\, \nabla\cdot\mathbf{n}\, dS_y, \quad (9)$$

which in fact is valid at all points $\mathbf{x}$, not just those on S. The reason for this is that when $\lambda = 1$ the flow, which is governed everywhere by the *same* set of equations, equals the sum of the imposed flow at infinity and a flow generated by a 'membrane' of point forces (Stokeslets) acting on the fluid, with no surface boundary conditions to be satisfied. This simplification when $\lambda = 1$ has also been exploited in the two-dimensional work of Buckmaster & Flaherty (1973).

Properties of the integral equation

If $\mathbf{u}$ (or equivalently the time scale t) is rescaled by $\gamma/2\pi\mu(1+\lambda)$, and $\mathbf{x}$ by a, then putting $u_i^\infty = E_{ij}x_j$ in (8) gives

$$u_i(\mathbf{x})+\frac{2(1-\lambda)}{1+\lambda}\int K_{ijk}u_j n_k dS = F_i(\mathbf{x}) \equiv \Omega E_{ij}x_j - \frac{1}{2}\int J_{ij}n_j \nabla.\mathbf{n}\, dS, \tag{10}$$

where $\Omega = 4\pi G\mu a/\gamma$, with G a characteristic strain rate of the imposed flow and a the radius of a spherical drop of the same volume as the drop undergoing deformation. $\mathbf{F}$ is the forcing term due to the imposed flow and the drop surface tension. The scaling of $\mathbf{u}$ reflects the fact that the time scale for deformation is determined by the larger of the viscosities μ and $\lambda\mu$. Clearly, there are two dimensionless parameters for the problem: λ and Ω, the latter being the ratio of the flow forces tending to increase the deformation of the drop to the surface-tension forces inhibiting it.

Given λ and Ω, (10) is an integral equation of the second kind for the surface velocities $\mathbf{u}(\mathbf{x})$. Difficulties arise whenever the operator on the left-hand side of (10) has neutral eigensolutions $\mathbf{u}_e$. As discussed by Ladyzhenskaya (1969, pp. 61, 62), there is just one such eigensolution when $\lambda = 0$ (which involves a volume change), and if $\lambda = \infty$ the six rigid-body motions for the drop are all eigensolutions. Away from these extremes, the operator is clearly non-singular at $\lambda = 1$ (and therefore in some neighbourhood of $\lambda = 1$), and since in practice no numerical difficulties were encountered lsewhere it appears most probable that it remains non-singular for $0 < \lambda < \infty$.

Axisymmetric drops

When the drop shape and the imposed flow are axisymmetric, the surface integrals in (10) can be reduced to line integrals by performing the azimuthal integrations analytically. In addition, for a linear extensional flow, the problem can be further simplified by exploiting the fore-and-aft symmetry of the drop. We thus suppose that the applied flow is given by $\mathbf{E} = \mathrm{diag}(2, -1, -1)$ and that the shape is defined by $r = R(x)\ (0 < x < l)$ with $R(-x) = R(x)$ and $R(l) = 0$, as shown in figure 2. With $\mathbf{u}(\mathbf{x}) = (u_r(x), u_x(x))$ and $\mathbf{F}(\mathbf{x}) = (F_r(x), F_x(x))$, (10) may then be written as

$$\begin{pmatrix} u_r(x) \\ u_x(x) \end{pmatrix} + \frac{2(1-\lambda)}{1+\lambda}\int_{-l}^{l}\begin{pmatrix} K_{rr} & K_{rx} \\ K_{xr} & K_{xx} \end{pmatrix}\begin{pmatrix} u_r(y) \\ u_x(y) \end{pmatrix} dy = \begin{pmatrix} F_r(x) \\ F_x(x) \end{pmatrix}, \tag{11}$$

where the kernel functions K have been determined explicitly in terms of complete elliptic integrals by Youngren & Acrivos (1975). In addition,

$$\begin{pmatrix} F_r(x) \\ F_x(x) \end{pmatrix} = \Omega\begin{pmatrix} -R(x) \\ 2x \end{pmatrix} - \frac{1}{2}\int_{-l}^{l}\begin{pmatrix} J_r(x-y) \\ J_x(x-y) \end{pmatrix}\left\{\frac{R''}{[1+R'^2(y)]^{\frac{3}{2}}} - \frac{1}{R[1+R'^2(y)]^{\frac{1}{2}}}\right\} dy, \tag{12}$$

where J_r and J_x have similarly been determined (Youngren & Acrivos 1975).

The fore-and-aft symmetry provides the further small simplification that (u_r, F_r) are even in x while (u_x, F_x) are odd. This observation means that, in the finite-difference form for the equations discussed in § 3, the number of unknown velocities is halved and, more important, that the $\lambda = \infty$ eigensolutions (rigid-body motions) of (10) are analytically excluded, thereby rendering the modified integral operator to be inverted non-singular at $\lambda = \infty$.

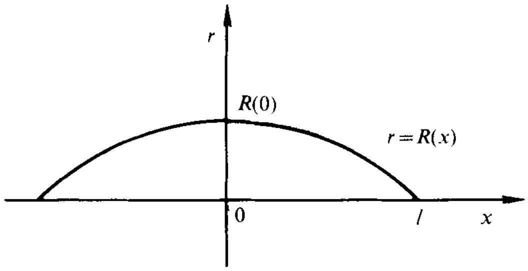

FIGURE 2. Co-ordinate system for an axisymmetric, fore-and-aft symmetric drop.

3. Numerical technique

The following numerical method was used. Starting from the given initial shape, $N+1$ surface points $\mathbf{x}_i$ $(i = 0, \ldots, N)$ spaced along the half-length, with $\mathbf{x}_0 = (0, R(0))$ and $\mathbf{x}_N = (l, 0)$, were chosen. A centrally centred finite-difference scheme was used to compute the slope and the curvature at each point. (In view of the infinite derivatives at the drop ends, this was found to be more convenient than the spline-fitting technique used by Youngren & Acrivos 1976.) On the assumption that $\mathbf{u}$ did not vary too rapidly over the surface, the integrals in (11) could be evaluated by a trapezoidal rule, with relative errors of order $|\mathbf{x}_i - \mathbf{x}_{i+1}|^2$, so that, symbolically,

$$\begin{pmatrix} u_r(x_i) \\ u_x(x_i) \end{pmatrix} + \begin{pmatrix} L_{rr}(x_i, x_j) & L_{rx}(x_i, x_j) \\ L_{xr}(x_i, x_j) & L_{xx}(x_i, x_j) \end{pmatrix} \begin{pmatrix} u_r(x_j) \\ u_x(x_j) \end{pmatrix} = \begin{pmatrix} F_r(x_i) \\ F_x(x_i) \end{pmatrix} \tag{13}$$

with summation over the x_j. A finite-difference form for the integrals in (12) was similarly obtained. Some care is required in determining certain contributions to these integrals since both $\mathbf{J}$ and $\mathbf{K}$ are singular when $x_i = x_j$. The singularity is logarithmic for $i = j \neq N$, but at the end, where $i = j = N$, the integral for J varies as $(l-y)^{-\frac{1}{2}}$. In both cases the integration over these segments was performed by subtracting the singularity, integrating it analytically and then adding the result to the numerically computed integral of the remainder. A check on the evaluation of $\mathbf{F}$ was made by increasing N and so decreasing the distance between collocation points. It was found that $N = 8$ was sufficient to give 2 % accuracy in the evaluation of J for nearly spherical shapes, but that for more extended drops with larger curvatures at their ends values of N up to 12 were required to maintain the same accuracy with a large density of points near the end.

In this finite-difference form, the integral equation is reduced to a matrix equation (13) for the velocities $u_r(x_i)$ and $u_x(x_i)$ $(i = 0, \ldots, N)$. This was solved by standard matrix-inversion techniques (Gaussian elimination), which encountered no difficulties away from the singular case $\lambda = 0$. Unfortunately, for $\lambda < 0{\cdot}1$, unacceptably large multiples of the undesired neutral eigensolution (at $\lambda = 0$) were found to appear. In addition, the resolution of the higher end curvatures was difficult for small λ, and the consequent smallness of the time step required for numerical stability (as discussed below) made it very difficult to construct reliable solutions by this method. Thus we shall present results only for $\lambda \geqslant 0{\cdot}3$.

After the velocity at each collocation point had been determined, the instantaneous normal component $u_n(x)$ was computed from

$$u_n(x) = (u_r - R'(x)\, u_x)/(1 + R'^2(x))^{\frac{1}{2}}$$

and, with a time step Δt, each point $\mathbf{x}_i$ on the surface was moved to $\mathbf{x}_i + u_n(x_i)\,\mathbf{n}(x_i)\,\Delta t$, with error $O(\Delta t^2)$. A new drop shape was thereby obtained. The process was then repeated until either an equilibrium was established with $u_n(x_i) \rightarrow 0$ for $i = 0, \ldots, N$, or the deformation appeared to increase without limit.

Numerical stability

For accurate evaluation of the integrals the typical spacing Δx between the x_i's must not be too large. In addition the following argument shows that, for numerical stability of the explicit scheme used here, the time step Δt must be correspondingly small. Suppose that at some stage in the evolution there is a small error ϵ in the computed position of the ith point. Then the consequent error in the curvature at that point will be $O(\epsilon/(\Delta x)^2)$ and, in the worst case $i = j = N$, the error in the computed velocity at x_j will be $O((\Delta x)^{\frac{1}{2}}\,\epsilon/(\Delta x)^2)$. Hence at the next time step the additional error in the position of the jth point will be $O(\epsilon \Delta t/(\Delta x)^{\frac{3}{2}})$. Thus if the error is not to increase, we must take

$$\Delta t \leqslant K(\Delta x)^{\frac{3}{2}}, \tag{14}$$

where K is an $O(1)$ constant. Numerical experiments with small values of N in fact demonstrated that $K = 1$ was sufficient.

A useful and simple check on the results as the time evolution progressed was to determine whether the incompressibility constraint

$$\int_S \mathbf{u}\,.\,\mathbf{n}\,dS = 0 \tag{15}$$

was automatically satisfied. In order to maintain the constraint to the desired accuracy it was found expedient to add extra points near the drop ends as the curvature there increased (and correspondingly to decrease the time step). For $\lambda < 0{\cdot}1$, however, unacceptably large volume changes occurred, no doubt associated with the appearance of the neutral eigensolution at $\lambda = 0$.

4. Numerical results

Solutions were obtained for $\lambda = 0{\cdot}3$, $0{\cdot}5$, 1, 2, 10 and 100. It was found that the results for $\lambda = 100$ were almost identical to those at $\lambda = 10$, thereby indicating that the steady-state solution which is found asymptotically as $\lambda \rightarrow \infty$ extends as far down as $\lambda = 10$. For each λ, the time evolution of the drop shape was computed for a range of values of Ω starting each time from a spherical initial condition. Two classes of behaviour were observed. For sufficiently strong ('supercritical') flows the drop would at first deform rapidly, then surface-tension forces would increase, so that the deformation proceeded less rapidly for a while, and finally, as the ends of the drop found themselves in a region of more rapidly moving fluid, the distortion would again increase faster. After further growth, an instability appeared which had a wavelength which was determined by the spacing of the collocation points. Changing either the number or spacing of these points, with the consequent change in Δt demanded by (14), altered the time of onset of the instability but could not ultimately suppress it.

On the other hand, for sufficiently weak ('subcritical') flows the normal velocity everywhere decreased monotonically from its initial value (proportional to Ω) to zero.

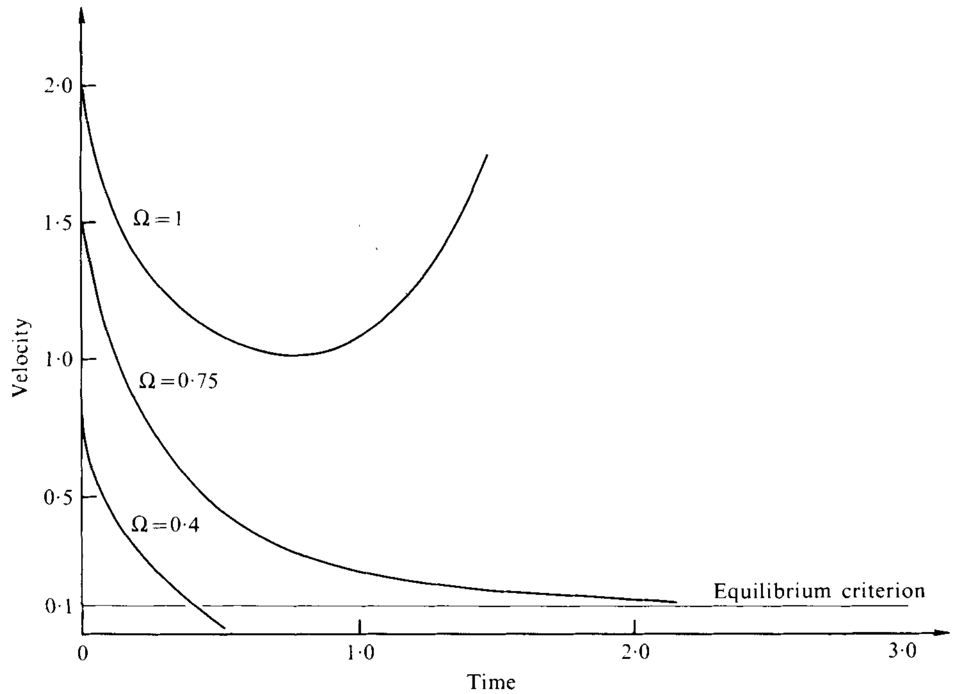

FIGURE 3. Velocity of fluid particle at end of drop in uniaxial extension as a function of time; $\lambda = 1$.

It was found that the velocity very rapidly became small along most of the length of the drop even for quite large Ω, and that subsequent deformation continued primarily at the ends, where further adjustment of the curvature was necessary.

A simple indication of this behaviour is given by figure 3, where the velocity of the point at the end of the drop is plotted as a function of time for various Ω's with $\lambda = 1$. Its initial value is 2Ω. The criterion applied for equilibrium to have been reached was that the non-dimensional normal velocity should not exceed 0·1 at every collocation point. It is seen from figure 3 that the insensitivity of the velocity minimum for values of Ω close to the critical value Ω_c poses a difficulty in determining Ω_c precisely. There is an uncertainty in Ω_c of about 0·03 from this source, and the computed value will in general be a slight overestimate. In comparing values of Ω_c at different λ, however, the inaccuracy from this source is reduced since the same criterion is applied.

An appropriate scalar measure of the magnitude of the deformation is

$$D_f = (L-B)/(L+B),$$

where L and B are respectively the lengths of the major and minor axes of the drop. Figure 4 shows graphs of

$$D_f(\Omega,\lambda) - \frac{3}{4\pi}\frac{19\lambda+16}{16\lambda+16}\Omega$$

for those cases where an equilibrium was attained. This quantity vanishes according to the linear theory of Taylor (1932), so the plot measures deviations from the linear theory. The results from the $O(D_f^2)$ theory of Barthès-Biesel & Acrivos (1973) are

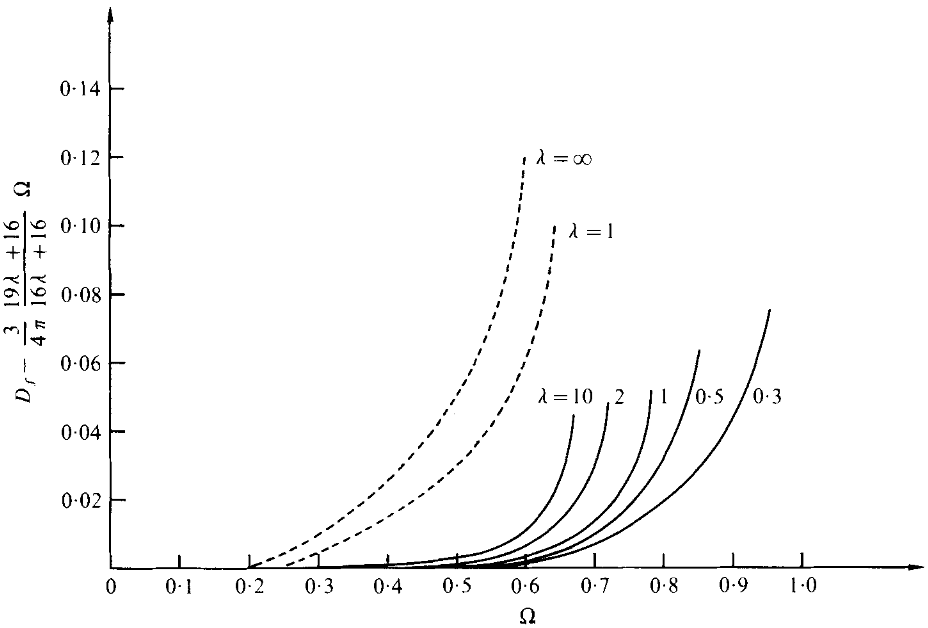

FIGURE 4. Equilibrium deformations $D_f - \frac{3}{4\pi}\frac{19\lambda+16}{16\lambda+16}\Omega$ as functions of Ω at a given λ. ——, numerical solution; – – –, quadratic theory (Barthès-Biesel & Acrivos 1973).

shown in figure 4. Figure 5 gives the critical flow strength Ω_c as a function of the viscosity ratio λ. Here the comparison is with the theories of Barthès-Biesel & Acrivos (1973) and with the large deformation, small λ, theory of Taylor (1964) and of Acrivos & Lo (1978).

Figure 4 shows that the linear, small deformation, theory is remarkably accurate in predicting the equilibrium distortion of a drop in uniaxial straining, even for flow rates comparable to those at which it will burst. The quadratic terms as computed by Barthès-Biesel & Acrivos (1973) have a very small range of usefulness, and indeed produce a somewhat worse estimate for $D(\Omega)$ than the linear theory over most of the range. On the other hand, the prediction of $\Omega_c(\lambda)$ from the quadratic terms is better than one might expect (see figure 5); the qualitative trend is correct, and indeed the numerical agreement is within 20 % for the range of λ considered here.

The Taylor (1964) asymptote, $\Omega_c(\lambda) = 0{\cdot}930\lambda^{-\frac{1}{6}}$ as $\lambda \to 0$, is derived from slender-body theory. The agreement with our results shown in figure 5 is again gratifyingly close, considering that the largest aspect ratio for our data is a little greater than 2. In addition, the infinitesimal stability analysis of Acrivos & Lo (1978) demonstrates that $\Omega > 0{\cdot}930\lambda^{-\frac{1}{6}}$ is a sufficient condition for instability (and hence burst) but leaves open the possibility that a finite amplitude instability (such as that involved in discretizing the equations, for instance) might produce a lower asymptotic value for Ω_c.

The mechanism of the final breakup of cylindrical drops in extension has been discussed by Tomotika (1936) and by Mikami, Cox & Mason (1975). They demonstrated that disturbances with a range of wavelengths along the drop will be amplified for sufficiently extended shapes, but that since the wavelengths of such disturbances continually increase owing to the stretching of the basic shape, the overall magnifica-

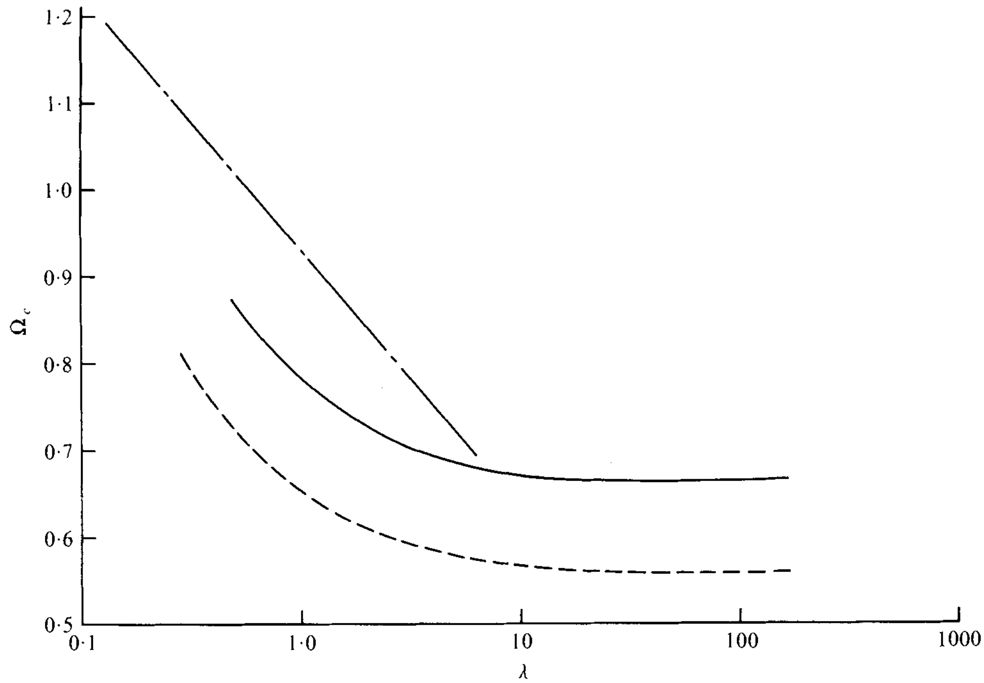

FIGURE 5. Critical value of Ω for bursting as a function of λ. ——, numerical solution; – – –, quadratic theory (Barthès-Biesel & Acrivos 1973); –·–·–, slender-body theory (Taylor 1964; Acrivos & Lo 1978).

tion of any one disturbance is finite. It appears then that on theoretical grounds there is no uniquely specified 'most unstable disturbance'. The appearance in our work of a numerical instability whose wavelength varies with the positions of the collocation points for sufficiently extended drops is in accord with this analysis.

A particular advantage of solving an initial-value problem for the shape was as done here is that, when there is a possibility that solutions will bifurcate, a specific prediction is made for the branch of the solution to be expected. The previous discussion has been exclusively concerned with the physically natural initial condition of a spherical shape, but the question remains as to the behaviour of the solution for other initial conditions. This has not been exhaustively nor systematically studied here. One case of particular importance and relevance to theoretical suspension studes with deformable micro-structures, however, is that in which a weak flow is applied to a drop which has been extended by a strong flow. Provisional numerical experiments indicated that two possibilities could arise: very extended initial shapes could be further extended to the point of bursting by a velocity field which would be subcritical with a spherical initial shape; less extended critical shapes could relax back to the equilibrium found previously. At any rate, in our studies no 'new' equilibria were found by this technique.

We have thus demonstrated that our method of solution leads to reliable results for deformation and breakup provided that λ is not too small. In principle, it should be possible of course to compute steady shapes for $\lambda < 0{\cdot}3$ by employing an implicit (in time) numerical technique which could be made significantly more stable, in a numerical sense, than the explicit procedure used here, although, clearly, such a calculation would not accurately simulate the transient deformation of the drop. In addition, it should be possible to devise a scheme which would incorporate the incompressibility constraint (15) in the formulation of the mathematical system to be solved. These extensions will, however, be relegated to a later study.

This work was supported in part by the National Science Foundation under grant ENG-23229, and by NATO Research Grant No. 1442. It was initiated at Cambridge University while A. Acrivos was on sabbatical under a Guggenheim Fellowship.

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